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Quantum Transition Amplitudes for Ergodic and for Completely Integrable Systems*

STEVEN ZELDITCH

Johns Hopkins University, Baltimore, Maryland 21218
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INTRODUCTION

Let (M, g) be a compact, riemannian manifold, let Δ be its Laplacian and let $\{\varphi_j\}$ denote a fixed orthonormal basis of Laplace eigenfunctions: $\Delta\varphi_j = -\lambda_j\varphi_j$, $0 = \lambda_0 < \lambda_1 \leq \lambda_2, \dots \uparrow \infty$. Various theorems relate the asymptotic properties of the λ_j 's and φ_j 's, as $\lambda_j \rightarrow \infty$, with the dynamical properties of the geodesic flow G^t on S^*M . Among the best-known results of this kind are the theorems of Duistermaat, Guillemin, Ivrii, Helton, and Colin de Verdiere, which relate the asymptotic distribution of the eigenvalues (mod 1) to the nature of the closed geodesics (M, g) . The gist of these is that the eigenvalues cluster around an arithmetic progression if all geodesics are closed, are somewhat more regularly distributed if at least one geodesic is not closed (the differences $\sqrt{\lambda_i} - \sqrt{\lambda_j}$ are dense in \mathbb{R}), and are asymptotically uniformly distributed (mod 1) if the set of closed geodesics has measure zero.

It is less well known that strong relationships also exist between the asymptotic behavior of Laplace eigenfunctions (as $\lambda_j \rightarrow \infty$) and the dynamical properties of the corresponding geodesic flows. A simple (but important) example of this is afforded by flat tori \mathbb{R}^n/L . Here the eigenfunctions are $\varphi_\lambda = e^{2\pi i \langle \lambda, \cdot \rangle}$ ($\lambda \in L^*$). To study the asymptotic behavior of the φ_λ , we form the matrix elements $(A\varphi_\lambda, \varphi_\lambda)$ with A a ΨDO of order 0. A simple computation shows that as $n \rightarrow \infty$, $(A\varphi_{n\lambda}, \varphi_{n\lambda}) \rightarrow \int_{T_\lambda} \sigma_A d\mu_\lambda$, where T_λ is the invariant torus for G^t consisting of unit vectors pointing in the direction $\lambda/|\lambda|$, and where $d\mu_\lambda$ is the unit mass translation invariant measure on T_λ . Thus the sequence $\{\varphi_{n\lambda}\}$ concentrates microlocally on $T_\lambda \in S^*M$.

A second example of this kind is given by the highest weight spherical harmonics $\{Y'_l\}$ on S^2 . A direct calculation here shows that (AY'_l, Y'_l) , for

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a given ΨDO , tends (as $l \rightarrow \infty$) to the integral of σ_A around the equatorial geodesic.

In both of these examples, the geodesic flow was completely integrable, and we saw that certain sequences of eigenfunctions tend, in some sense, to δ -functions along invariant tori. A rather opposite type of theorem was stated by A. I. Snirelman in 1974: if G' is ergodic, then eigenfunctions tend toward uniform distribution in S^*M in the sense that $(A\varphi_j, \varphi_j) \rightarrow \int_{S^*M} \sigma_A d\mu$ ($d\mu$ = Liouville measure). (A complete proof, together with fuller historic and heuristic remarks, may be found in [16, 1].)

Our purpose in this paper is to enlarge on the theme that dynamical assumptions on G' have a strong impact on the asymptotic behavior of eigenfunctions. In particular, we will see what happens when G' is mixing, and, at the opposite extreme, when G' is completely integrable.

Before describing our results, let us explain more fully what we mean by the asymptotic behavior of eigenfunctions. The idea is always to "test" the eigenfunctions against the 0th order ΨDO 's on $L^2(M)$, and to see what kinds of limits they have as the eigenvalue tends to infinity. "Testing" here means to form the matrix coefficients $(A\varphi_i, \varphi_j)$ of the ΨDO A relative to the $\{\varphi_j\}$. Such a coefficient is interpreted in quantum mechanics (roughly) as the probability amplitude that a free particle in the energy state ϕ_i makes a transition to ϕ_j while A is being observed, or, equivalently, as the expected value of A during a transition from the state ϕ_i to the state ϕ_j . The correspondence principle of quantum mechanics suggests that, as the energy tends to ∞ , such transition amplitudes should have a classical limit. To make this precise, we will view the matrix element $(A\varphi_i, \varphi_j)$ as a linear functional of the symbol σ_A of A . Thus, we fix once and for all an association $\sigma \rightarrow \text{Op}(\sigma)$ of 0th order ΨDO 's to symbols $\sigma \in C^\infty(S^*M)$ (the results will be independent of this choice). We then define the distribution $d\Phi_{i,j} \in \mathcal{D}'(S^*M)$ by

$$\langle \sigma, d\Phi_{i,j} \rangle \stackrel{\text{def}}{=} (\text{Op}(\sigma) \varphi_i, \varphi_j). \quad (0.1)$$

The problem of describing the asymptotic behavior of the φ_j can now be stated precisely: determine the weak limits of the $\{d\Phi_{i,j}\}$.

The fact that these weak limits have something to do with the geodesic flow has been known for a long time. In the mathematics literature, for instance, it is noted in [15, 5] that the weak limits of the diagonal elements $\{d\Phi_{i,i}\}$ are invariant (probability) measures for G' . In special cases, these can be determined by symmetry considerations, or by direct computation (as with the torus and sphere examples above). In general, however, it appears to be a difficult problem to determine which invariant measures μ show up as such classical limits of eigenfunctions.

The present paper contains several kinds of new results on this problem. The first is an extension of the result described above, to the effect that

$d\Phi_{j,j}$ almost certainly (in j) tends to Liouville measure if G' is ergodic. The extension is to describe the limits of sequences of off-diagonal elements $\{d\Phi_{i,j}, i \neq j\}$. This may seem at first to be a rather dull modification. However, it turns out to have more structure than might first be suspected. To begin with, a sequence $\{d\Phi_{i_r,j_r}\}$ with a unique limit can occur only if $\sqrt{\lambda_{i_r}} - \sqrt{\lambda_{j_r}}$ tends to a limiting gap α , and then the limit is an eigenmeasure of G' of eigenvalue $e^{i\alpha t}$ (a good example of such an eigenmeasure to keep in mind is a fourier coefficient along an orbit). In the case where $\lambda_{i_r} = \lambda_{j_r}$, i.e., where one has a large sequence of multiple eigenvalues, the limit measure must therefore be invariant. Our first result is that if G' is ergodic, and if a positive proportion of the λ_j 's is multiple, then, up to a sparse subsequence (of "proportion zero"), the $d\Phi_{i_r,j_r}$'s with $i_r \neq j_r$ but $\lambda_{i_r} = \lambda_{j_r}$ must tend to zero. (Compare this with the situation for the diagonal elements.)

Our second result concerns the limits when the limiting gap $\alpha \neq 0$. For this we need to assume that the flow G' is mixing. In that case one finds again that the off-diagonal elements tend almost certainly to 0.

The proofs of these results involve the theorem cited above on limits of diagonal elements. They require as well a new ingredient: a kind of L^2 -ergodic theorem for matrix element of ψD^0 's relative to Laplace eigenfunctions (Lemma A). Some explanation of why they are true is afforded by a description of the possible (non-invariant) eigenmeasures for hyperbolic, ergodic flows G' : for these eigenmeasures are all singular relative to Liouville measure, and consequently "shouldn't" arise as limits of "fat" (i.e., positive density) sequences of $d\Phi_{i_r,j_r}$'s (see Proposition C).

Our final results (in Section 3) concern the asymptotic behavior of eigenfunctions in the opposite extreme case of completely integrable geodesic flow. Here, there is a natural choice of orthonormal basis which reflects the foliation of S^*M onto invariant torii for G' . Indeed, such a basis comes out of the "ladder" theory of Guillemin and Sternberg for quantizations of Hamiltonian toral actions. This section was partly inspired by an (unpublished) article of A. Uribe, which uses the ladder theory to show that diagonal sequences corresponding to such ladder eigenfunctions tend to δ -functions on invariant tori. Its other inspiration was Colin de Verdiere's work on microlocal normal forms for commuting operators in the completely integrable case [1]. Essentially, this allows us to reduce the problem to the simple case, described above, of flat tori. It seems to give somewhat sharper results than does the ladder theory (see Theorem E).

1. CLASSICAL LIMITS FOR COHERENT FAMILIES

We are interested in the value limits of the family $\{d\Phi_{i,j}\}$ of distributions on $C^\infty(S^*M)$. It is natural to make the definitions:

DEFINITION 1.1. (a) A subset $\mathcal{F} = \{d\Phi_{i_r, j_r} : r, s \in N^+\}$ of $\{d\Phi_{i, j}\}$ is a *coherent family* if it has a unique vague limit point.

(b) The eigenvalue pairs $\{(\mu_{i_r}, \mu_{j_r})\}$, with $\mu_j = \sqrt{\lambda_j}$, form the *spectrum of the family*, to be denoted $\text{spec}(\mathcal{F})$.

(c) If the vague limit of a coherent family is 0, it is called a *vanishing family*, otherwise, *non-vanishing*.

We have:

PROPOSITION 1.1. Suppose $\mathcal{F} = \{d\Phi_{i_r, j_r}\}$ is a coherent non-vanishing family. Then:

(i) the differences $\{\mu_{i_r} - \mu_{j_r}\}$ of pairs in $\text{spec}(\mathcal{F})$ have a unique limit point α , and

(ii) the vague limit of the family is an eigenmeasure for G^t of eigenvalue $e^{i\alpha t}$.

Proof. Let

$$U(t) = \exp(it\sqrt{-\Delta}). \quad (1.1)$$

Then for any zeroth order $\psi D0$ A ,

$$(U(-t)AU(t)\Phi_{i_r}, \Phi_{j_r}) = e^{it(\mu_{i_r} - \mu_{j_r})}(A\Phi_{i_r}, \Phi_{j_r}) \quad (1.2)$$

and

$$(U(-t)AU(t)\Phi_{i_r}, \Phi_{j_r}) = (\text{Op}(\sigma_A \circ G^t)\Phi_{i_r}, \Phi_{j_r}) + (R_t\Phi_{i_r}, \Phi_{j_r}), \quad (1.3)$$

where R_t is a $\psi D0$ of order -1 (Egorov Theorem). Viewing σ_A as a function on S^*M extended by homogeneity (of order 0) to \tilde{T}^*M , we have, for $\sigma \in C^\infty(S^*M)$,

$$\begin{aligned} e^{it(\mu_{i_r} - \mu_{j_r})} \langle \sigma, d\Phi_{i_r, j_r} \rangle &= \langle \sigma \circ G^t, d\Phi_{i_r, j_r} \rangle \\ &+ O(\min\{|\mu_{i_r}|^{-1}, |\mu_{j_r}|^{-1}\}). \end{aligned} \quad (1.4)$$

Coherence plus non-vanishing clearly implies that $\{\mu_{i_r} - \mu_{j_r}\}$ has a unique limit, say α . Further, the vague limit of the family must be an eigen-distribution of eigenvalue $e^{i\alpha t}$. A standard argument from the diagonal case shows then that every limit is a measure: namely, any limit $d\nu$ of $(A\Phi_{i_r}, \Phi_{j_r})$ is equally one of $((A+K)\Phi_{i_r}, \Phi_{j_r})$ for any compact operator K . So $|(\sigma_A, d\nu)| \leq \inf\{\|A+K\| : K \text{ compact}\}$. But the right side is $\sup_{\tilde{T}^*M} |\sigma_A|$, so $d\nu$ is a measure [15, 5]. ■

We can also carry over the notion of the density of a limit measure [2, 16]:

DEFINITION 1.2. (a) $N(\lambda) = \# \{ \mu_j \leq \lambda \}$.

(b) If $\mathcal{F} = \{ d\Phi_{i_r, j_r} \}$ is a coherent family, let $N(\lambda, \mathcal{F}) = \# \{ (\mu_{i_r}, \mu_{j_r}) : \mu_{i_r}, \mu_{j_r} \leq \lambda \}$. Further, let the density of the family be $D^*(\mathcal{F}) \stackrel{\text{def}}{=} \lim_{\lambda \rightarrow \infty} (N(\lambda, \mathcal{F})/N(\lambda))$.

PROPOSITION 1.2. Let $\mathcal{F} = \{ d\Phi_{i_r, j_r} \}$ be a coherent non-vanishing family. Then $D^*\{d\Phi_{i_r, j_r}\} < \infty$.

Proof. Suppose $\{d\Phi_{i_r, j_r}\} \rightarrow d\mu$. Since $d\mu \neq 0$, $\exists \varepsilon > 0$ and $\sigma \in C^\infty(S^*M)$ such that $|\langle \sigma, d\mu \rangle| > \varepsilon$. Hence $\exists \lambda_0$ such that $\lambda \geq \lambda_0$ implies

$$\frac{1}{N(\lambda, \mathcal{F})} \sum_{\mu_{i_r}, \mu_{j_r} \leq \lambda} |(\text{Op}(\sigma) \Phi_{i_r}, \Phi_{j_r})|^2 > \varepsilon. \quad (1.5)$$

On the other hand, a limit formula due to Widom asserts that

$$\frac{1}{N(\lambda)} \sum_{\mu_i, \mu_j \leq \lambda} |(\text{Op}(\sigma) \Phi_i, \Phi_j)|^2 \rightarrow \frac{1}{\text{vol}(S^*M)} \int_{S^*M} |\sigma|^2 d\omega. \quad (1.6)$$

Indeed, let us define for any 0th order $\psi D0 B$,

$$\text{tr}_A B \stackrel{\text{def}}{=} \lim_{\lambda \rightarrow \infty} \frac{1}{N(\lambda)} \sum_{\mu_i \leq \lambda} (B\Phi_i, \Phi_i). \quad (1.7)$$

Then (cf. [5])

$$\text{tr}_A B = \frac{1}{\text{vol}(S^*M)} \int_{S^*M} \sigma_B d\omega. \quad (1.8)$$

Further, let π_λ denote orthoprojection onto $\mathcal{H}_\lambda = \{\text{span}\{\Phi_j : \mu_j \leq \lambda\}\}$. Then

$$\sum_{\mu_i, \mu_j \leq \lambda} |(\text{Op}(\sigma) \Phi_i, \Phi_j)|^2 = \text{tr}(\pi_\lambda \text{Op}(\sigma) \pi_\lambda)^* (\pi_\lambda \text{Op}(\sigma) \pi_\lambda). \quad (1.9)$$

Hence (1.5) follows as long as

$$\text{tr} \pi_\lambda \text{Op}(\sigma)^* (1 - \pi_\lambda) \text{Op}(\sigma) \pi_\lambda = o(N(\lambda)). \quad (1.10)$$

We refer to [5] for the proof of this.

Combining (1.5) and (1.6) we get

$$\varepsilon \frac{N(\lambda, \mathcal{F})}{N(\lambda)} \leq \frac{1}{N(\lambda)} \sum_{\mu_i, \mu_j \leq \lambda} |(A\Phi_i, \Phi_j)|^2 \rightarrow \|\sigma_A\|_{L^2(S^*X)}^2, \quad (1.11)$$

proving the proposition. ■

This proposition shows why density is defined using $N(\lambda)$ rather than $N(\lambda)^2 = \#\{\mu_i, \mu_j \leq \lambda\}$. Coherent off-diagonal families behave in a way parallel to diagonal families; in particular, in regard to their size.

To enlarge on this, consider the impact of Proposition 1.1(i) on the estimate of $N(\lambda, \mathcal{F})$. By 1.1(i), the possible second components μ_j of pairs (μ_i, μ_j) in $\text{Spec}(\mathcal{F})$ with fixed μ_i must lie in an interval around μ_i with radius tending to zero as $\mu_i \rightarrow \infty$. However, 1.1(i) gives no estimate on the rate of decay of these intervals, nor of the number of μ_j 's in these intervals which actually contribute to $\text{Spec}(\mathcal{F})$. Hence, 1.1(i) alone can't rule out that all μ_j in the intervals of essentially fixed radius δ around μ_i contribute to \mathcal{F} . The asymptotic number of such eigenvalues is calculated in [4] under generic assumptions on (M, g) : the upshot is that if the set of closed geodesics has measure 0, then $\#\{(\mu_i, \mu_j) : \mu_i, \mu_j \leq \lambda, |\mu_i - \mu_j| < \delta\} \sim \text{vol}(M) \delta \lambda^{n-1} N(\lambda)$ ($n = \dim M$). In our problem, δ is tending to zero as $\lambda \rightarrow \infty$, but not by any definite rate. Thus, Proposition 1.2 improves on Proposition 1.1(i) by a factor (essentially) of λ^{n-1} .

The only exception to Proposition 1.2 is given by a coherent vanishing family. This is a large exception:

PROPOSITION 1.3. *There is a vanishing subfamily $\mathcal{F}^0 = \{d\Phi_{i,j}^0\}$ with $N(\lambda, \mathcal{F}^0)/N(\lambda)^2 \rightarrow 1$.*

Proof. Immediately from (1.6) we see that $(1/N(\lambda)^2) \sum_{\mu_i, \mu_j \leq \lambda} |(A(\psi_i, \psi_j))|^2 \rightarrow 0$. Since the terms are positive it follows by a standard argument [14] that after removing a subset of the terms from $\{\mu_i, \mu_j \leq \lambda\}$ of proportion tending to 0, all limit points of the remaining terms are zero. These terms apparently depend on A , but a kind of diagonalization argument shows that there is a density 1 subset which tends to 0 for all A (cf. [16, 3]). ■

Despite Proposition 1.3, there can be "fat" coherent families (as "fat," at any rate, as a coherent diagonal family). The basic problems on these families are analogous to the diagonal case: What are the coherent families and their spectra, how "fat" are they (density relative to $N(\lambda)$), and what are their limits? The rest of this paper gives partial answers to these questions for special geodesic flows.

2. ERGODIC GEODESIC FLOW

In [11, 16, 1] it is proved that when G' is ergodic there is a coherent diagonal family of density 1 whose classical limit is Liouville measure. One conjectures that the full diagonal family is coherent this way: essentially because the other candidates for classical limits are "unstable." Our theme

in this section is in the same spirit: coherent off-diagonal families should be vanishing. When G' is only assumed to be ergodic, this theme is borne out only partially by Theorem A.

However, when we further assume G' to be mixing, we will show (Theorem A2) that any coherent family of positive density is vanishing. The main example of a mixing geodesic flow comes from compact, negatively curved manifolds. These are in fact hyperbolic (Anosov). For such ergodic and hyperbolic geodesic flows, one can determine explicitly all finite non-invariant eigen measures (following K. Sigmund). They are all given by "orbital fourier coefficients" along exceptional, unstable orbits. Thus, our theme will be substantiated for these cases; one conjectures that for such G' all coherent off-diagonal families are vanishing.

We begin with a kind of L^2 ergodic theorem:

LEMMA A. *Let G' be ergodic. Then for any 0th order ψDO A one has*

$$(\forall \varepsilon \exists \delta) \lim_{\lambda \rightarrow \infty} \frac{1}{N(\lambda)} \sum_{\substack{\mu_i, \mu_j \leq \lambda \\ i \neq j \\ |\mu_i - \mu_j| < \delta}} |(A\Phi_i, \Phi_j)|^2 < \varepsilon.$$

Proof. Let us write

$$\overline{\sigma_A} = \frac{1}{\text{vol}(S^*M)} \int_{S^*M} \sigma_A d\omega, \quad (2.1i)$$

$$\|\sigma_A\|_2^2 = \frac{1}{\text{vol}(S^*M)} \int_{S^*M} |\sigma_A|^2 d\omega. \quad (2.1ii)$$

We recall from (1.7), (1.8) that

$$\text{tr } {}_A A = \overline{\sigma_A} \quad (2.2i)$$

$$\frac{1}{N(\lambda)} \sum_{\mu_i, \mu_j \leq \lambda} |(A\Phi_i, \Phi_j)|^2 \rightarrow \|\sigma_A\|_2^2. \quad (2.2ii)$$

Now consider the averaged ψDO 's [6]:

$$A_f = \int_{-\infty}^{\infty} \hat{f}(t) U(-t) A U(t) dt, \quad (2.3)$$

where \hat{f} is bounded and compactly supported. Then by Egorov

$$\sigma_{A_f}(x, \xi) = \int_{-\infty}^{\infty} \hat{f}(t) \sigma_A(G'(x, \xi)) dt. \quad (2.4)$$

From (2.2ii) we get

$$\begin{aligned} \frac{1}{N(\lambda)} \sum_{\mu_i, \mu_j \leq \lambda} |(A\Phi_i, \Phi_j)|^2 |f(\mu_i - \mu_j)|^2 \\ \rightarrow \left\| \int_{-\infty}^{\infty} \hat{f}(t) \sigma_A(G'(x, \xi)) dt \right\|_2^2. \end{aligned} \quad (2.5)$$

Let $\hat{f}_T(t) = (1/2T) 1_{[-T, T]}$ (characteristic function). Then (2.5) reads

$$\begin{aligned} \frac{1}{N(\lambda)} \sum_{\mu_i, \mu_j \leq \lambda} |(A\Phi_i, \Phi_j)|^2 \left| \frac{\sin T(\mu_i - \mu_j)}{T(\mu_i - \mu_j)} \right|^2 \\ \rightarrow \left\| \frac{1}{2T} \int_{-T}^T \sigma_A(G'(x, \xi)) dt \right\|_2^2. \end{aligned} \quad (2.6)$$

By [16, 3] one has

$$\frac{1}{N(\lambda)} \sum_{\mu_j \leq \lambda} |(A\Phi_j, \Phi_j) - \overline{\sigma}_A|^2 \rightarrow 0. \quad (2.7)$$

Combining (2.2i), (2.2ii), and (2.7) we have

$$\frac{1}{N(\lambda)} \sum_{\mu_j \leq \lambda} |(A\Phi_j, \Phi_j)|^2 \rightarrow |\overline{\sigma}_A|^2. \quad (2.8)$$

Putting together (2.6) and (2.8)

$$\begin{aligned} \frac{1}{N(\lambda)} \sum_{\substack{\mu_i, \mu_j \leq \lambda \\ i \neq j}} |(A\Phi_i, \Phi_j)|^2 \left| \frac{\sin T(\mu_i - \mu_j)}{T(\mu_i - \mu_j)} \right|^2 \\ \rightarrow \left\| \frac{1}{2T} \int_{-T}^T \sigma_A(G'(x, \xi)) dt \right\|_2^2 - |\overline{\sigma}_A|^2. \end{aligned} \quad (2.9)$$

By the L^2 ergodic theorem, the right side tends to 0 as $T \rightarrow \infty$. Thus

$$(\forall \varepsilon)(\exists T) \lim_{\lambda \rightarrow \infty} \frac{1}{N(\lambda)} \sum_{\substack{\mu_i, \mu_j \leq \lambda \\ i \neq j}} |(A\Phi_i, \Phi_j)|^2 \left| \frac{\sin T(\mu_i - \mu_j)}{T(\mu_i - \mu_j)} \right|^2 < \varepsilon. \quad (2.10)$$

In particular $\sin x/x > 1/2$ if $|x| < 1/2$ (say). So

$$(\forall \varepsilon \exists T) \lim_{\lambda \rightarrow \infty} \frac{1}{N(\lambda)} \sum_{\substack{\mu_i, \mu_j \leq \lambda \\ i \neq j \\ |\mu_i - \mu_j| < 1/2T}} |(A\Phi_i, \Phi_j)|^2 < \varepsilon. \quad \blacksquare \quad (2.11)$$

We now give our main theorem on coherent families when G^T is ergodic. To state it we need the notion of relative density. Let \mathcal{F} be any family $\{d\Phi_{i_r, j_r}\}$. We say $\mathcal{F}' \subset \mathcal{F}$ is a subfamily of relative density 1 if $\lim_{\lambda \rightarrow \infty} (N(\lambda, \mathcal{F}')/N(\lambda, \mathcal{F})) = 1$. We have:

THEOREM A. *Suppose G^t is ergodic, and let \mathcal{F} be any off-diagonal family so that the only limit point of $\{\mu_{i_r} - \mu_{j_s}\} = 0$, i.e., so that all its limit measures are invariant. If $D^*(\mathcal{F}) > 0$, then there exists a coherent subfamily $\mathcal{F}' \subset \mathcal{F}$ of relative density 1 which is vanishing.*

Proof. Suppose \mathcal{F} is such a family. \mathcal{F} has a coherent vanishing subfamily \mathcal{F}' of relative density 1 if and only if

$$\frac{1}{N(\lambda, \mathcal{F})} \sum_{\mu_{i_r}, \mu_{j_r} \leq \lambda} |(A\Phi_{i_r}, \Phi_{j_r})|^2 \rightarrow 0. \quad (2.12)$$

Since $D^*(\mathcal{F}) > 0$, $1/N(\lambda, \mathcal{F}) \ll 1/N(\lambda)$. It therefore suffices to prove

$$\frac{1}{N(\lambda)} \sum_{\mu_{i_r}, \mu_{j_r} \leq \lambda} |(A\Phi_{i_r}, \Phi_{j_r})|^2 \rightarrow 0. \quad (2.13)$$

But the assumption that 0 is the only limit point of $\{\mu_{i_r} - \mu_{j_r}\}$ implies

$$\begin{aligned} & \lim_{\lambda \rightarrow 0} \frac{1}{N(\lambda)} \sum_{\mu_{i_r}, \mu_{j_r} \leq \lambda} |(A\Phi_{i_r}, \Phi_{j_r})|^2 \\ (\forall \delta > 0) & \leq \lim_{\lambda \rightarrow \infty} \frac{1}{N(\lambda)} \sum_{\substack{i \neq j \\ \mu_i, \mu_j \leq \lambda \\ |\mu_i - \mu_j| \leq \delta}} |(A\Phi_i, \Phi_j)|^2. \end{aligned} \quad (2.14)$$

So, (2.13), hence Theorem A, follows from Lemma A. ■

We next strengthen Theorem A by removing the hypothesis that 0 is the only limit of the family. However, we will then have to add the hypothesis that G^t is mixing.

THEOREM B. *Suppose G^t is mixing. Suppose further that $\mathcal{F} = \{d\Phi_{i_r, j_r}\}$ is an off-diagonal family with $D^*(\mathcal{F}) > 0$ and such that $\{\mu_{i_r} - \mu_{j_r}\}$ has exactly one limit point α . Then there exists a subfamily \mathcal{F}' of relative density 1 which is vanishing.*

Proof. The criterion that \mathcal{F} have such a vanishing subfamily \mathcal{F}' is again that

$$\lim_{\lambda \rightarrow \infty} \frac{1}{N(\lambda, \mathcal{F})} \sum_{\mu_{i_r}, \mu_{j_r} \leq \lambda} |(A\Phi_{i_r}, \Phi_{j_r})|^2 \rightarrow 0 \quad (2.15)$$

and it suffices to prove (2.15) with $N(\lambda)$ in place of $N(\lambda, \mathcal{F})$. In place of (2.14) we now have

$$\begin{aligned}
 & \lim_{\lambda \rightarrow 0} \frac{1}{N(\lambda)} \sum_{\mu_{i_r}, \mu_{j_r}} |(A\Phi_{i_r}, \Phi_{j_r})|^2 \\
 (\forall \delta > 0) \quad & \leq \lim_{\lambda \rightarrow \infty} \frac{1}{N(\lambda)} \sum_{\substack{\mu_i, \mu_j \leq \lambda \\ i \neq j \\ |\mu_i - \mu_j - \alpha| \leq \delta}} |(A\Phi_i, \Phi_j)|^2. \quad (2.16)
 \end{aligned}$$

To show the right side of (2.16) tends to zero as $\delta \rightarrow 0$, we replace $f_T(t)$ in (2.6) by $f_T(t - \alpha)$. Relation (2.6) becomes

$$\begin{aligned}
 & \frac{1}{N(\lambda)} \sum_{\substack{\mu_i, \mu_j \leq \lambda \\ i \neq j}} |(A\Phi_i, \Phi_j)|^2 \left| \frac{\sin T(\mu_i - \mu_j - \alpha)}{T(\mu_i - \mu_j - \alpha)} \right|^2 \\
 & \rightarrow \left\| \frac{1}{2T} \int_{-T}^T e^{ixt} \sigma_A(G'(x, \xi)) dt \right\|_2^2 - \left| \frac{\sin \alpha T}{\alpha T} \right|^2 |\bar{\sigma}_A|^2. \quad (2.17)
 \end{aligned}$$

The term $|\sin \alpha T / \alpha T|^2 |\bar{\sigma}_A|^2$ obviously tends to 0 when $\alpha \neq 0$, so we only need to prove the first one does too. However, let $\mu_{(x, \xi)}^\alpha$ denote any weak limit of the measures $\mu_{(x, \xi)}^\alpha, T(f) \stackrel{\text{def}}{=} (1/2T) \int_{-T}^T e^{ixt} f(G'(x, \xi)) dt$. It is clear that $\mu_{(x, \xi)}^\alpha$ must be an eigenmeasure of G' : $G'^* \mu_{(x, \xi)}^\alpha = e^{-ixt} \mu_{(x, \xi)}^\alpha$. But if G' is mixing, $\mu_{(x, \xi)}^\alpha$ must be zero for almost all (x, ξ) (relative to Liouville measure $d\omega$). Indeed, if the orbit through (x, ξ) is uniformly distributed relative to $d\omega$, then the total variation measure $|\mu_{(x, \xi)}^\alpha|$ (=the orbital average through (x, ξ)) is $d\omega$. By polar decomposition $\mu_{(x, \xi)}^\alpha = h d\omega$, $|h| = 1$ a.e. ($d\omega$). Evidently h is an eigenfunction of G' . Following a well-known argument [10], we consider $\mu_{(x, \xi)}^\alpha \otimes \overline{\mu_{(x, \xi)}^\alpha}$ on $C(S^*M \times S^*M)$. It is clearly invariant relative to $G' \times G'$. It follows that $h(x) \overline{h(y)}$ is an invariant bounded measurable function. But G' mixing implies $G' \times G'$ ergodic [10]. So $h(x) \overline{h(y)}$ must be constant a.e. ($d\omega \otimes d\omega$), and so h is constant a.e. ($d\omega$). Since h is an eigenfunction, the constant must be zero. Consequently, if (x, ξ) is a point whose orbit is uniformly distributed, and if $\alpha \neq 0$, then

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T e^{-ixt} f(G'(x, \xi)) dt$$

exists and equals zero (a.e.) for any $f \in C(S^*M)$. It follows that $\|(1/2T) \int_{-T}^T e^{-ixt} f(G'(x, \xi)) dt\|_2^2$ tends to zero for any f .

The remainder of the proof of Theorem B follows exactly as in Theorem A. ■

To expand on this a little, let us further suppose that G' is hyperbolic, i.e., that

$$T(S^*M) = E \oplus E^s \oplus E^u$$

is a splitting into invariant subbundles, with E a 1-dimensional bundle tangent to the flow, and where there are constants $C, \lambda > 0$ s.t.

$$\|dG'(v)\| \leq Ce^{-\lambda t} \|v\| \quad v \in E^s, t > 0$$

$$\|dG^{-t}(v)\| \leq Ce^{-\lambda t} \|v\| \quad v \in E^u, t > 0$$

($\|v\|$ some metric on $T(S^*M)$).

The best known example of a mixing geodesic flow is also hyperbolic (M compact, negatively curved). By slightly modifying a theorem of K. Sigmund [12], one can identify all eigenmeasures in the hyperbolic case. One has:

PROPOSITION C. *Suppose G' is a hyperbolic geodesic flow. Then*

(a) *each finite eigenmeasure is (up to scalars) an orbital fourier coefficient*

$$\mu_{z_0}^\alpha(f) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T e^{-i\alpha t} f(G^t(z_0)) dt$$

through some z_0 ;

(b) *if G' is also ergodic, then all non-invariant eigenmeasures are singular relative to Liouville measure.*

Proof. (a) Suppose ν is a finite eigenmeasure, $G'_* \nu = e^{i\alpha t} \nu$. Then its total variation measure $|\nu|$ is positive and invariant. According to [12], any invariant probability measure is an orbital average $\mu_{z_0}(f) = \lim_{T \rightarrow \infty} (1/2T) \int_{-T}^T f(G^t(z_0)) dt$ through some “quasi-regular” point z_0 (a point for which the limit exists for all $f \in C(S^*M)$). It follows that $|\nu| = \|\nu\| \mu_{z_0}$ for some z_0 , and by polar decomposition that $\nu = h \|\nu\| \mu_{z_0}$ for some $h \in L^1(\mu_{z_0})$, $|h| = 1$ a.e. Then $h(G^t(z_0)) = e^{i\alpha t} h(z_0)$, so $\nu(f) = \lim_{T \rightarrow \infty} (1/2T) \int_{-T}^T e^{i\alpha t} f(G^t(z_0)) dt$ up to a constant.

(b) Hyperbolic and ergodic flows are mixing so (b) follows as in the proof of Theorem B. ■

In sum, suppose G' is an ergodic and hyperbolic geodesic flow, and suppose \mathcal{F} is an off-diagonal coherent family. Then the classical limit of \mathcal{F} must be an orbital fourier coefficient along an exceptional orbit, at least up to a scalar multiple. It would be surprising if such an unstable limit could actually appear; one conjectures that the limit is 0. When $D^*(\mathcal{F}) > 0$, this

follows from Theorem B, at least for a subfamily of relative density 1. Actually, it is possible that a theorem of [2] to the effect that the density of a coherent diagonal family (a special kind of “quasi-mode”) is no greater than the mass of its microsupport (Liouville measure of the support of the limit measures of the quasi-mode) could be generalized to off-diagonal families. It would follow that only a density zero family could tend to a singular eigenmeasure. At any rate, the results of this section are exactly analogous to those for the diagonal case: one understands the behaviour of coherent families of positive density, but has no understanding of zero density families.

3. COMPLETELY INTEGRABLE GEODESIC FLOW

In this section we will construct many coherent, zero density families \mathcal{F} associated to invariant tori of certain completely integrable geodesic flows. The diagonal families we present here have been independently and previously constructed by A. Uribe in [13]. Our method is however closer to Colin de Verdiere’s in [1]; indeed our main result, Theorem E, follows very easily from Colin de Verdiere’s work.

We begin with a resume of the background material we will need from the Guillemin–Sternberg ladder theory, and from Uribe’s paper. We will use this to give a mean value version of the classical limit formula. We then give our main result as an application of Colin de Verdiere’s normal form theory.

DEFINITION 3.1. $\gamma: T^d \times Z \rightarrow Z$ is a homogeneous hamiltonian torus action if:

- (i) $T^d = (\mathbb{R}/2\pi\mathbb{Z})^d$,
- (ii) $Z = \dot{T}^*M$, $\dim M = d$,
- (iii) $(\gamma(e^{i\theta}, rz)) = r\gamma(e^{i\theta}, z)$, $r > 0$,

(iv) there exists an injective homomorphism from \mathfrak{t} ($=$ Lie algebra of T^d) to the Poisson algebra $C^\infty(Z)(\xi \rightarrow \phi^\xi)$ so that the following diagram commutes:

$$\begin{array}{ccc} & & C^\infty(Z) \\ & \nearrow & \downarrow \\ \mathfrak{t} & & \text{Vect}(Z) \end{array}$$

($\mathfrak{t} \rightarrow \text{Vect}(Z)$ is the derived action, which we write $\xi \rightarrow \xi_X$ and $C^\infty(Z) \rightarrow \text{Vect}(Z)$ takes $\phi \in C^\infty(Z)$ to its Hamilton vector field H_ϕ).

Associated to a Hamiltonian group action is its moment map Φ :

DEFINITION 3.2. The moment map $\Phi: Z \rightarrow \mathfrak{t}^*$ is $\langle \Phi(z), \xi \rangle = \phi^z(z)$, $\langle \cdot, \cdot \rangle$ being the pairing of $\mathfrak{t}^* \times \mathfrak{t} \rightarrow \mathbb{R}$.

In more concrete terms, let $\{\xi_1, \dots, \xi_d\}$ be a fixed basis for \mathfrak{t} , and let $a_i = \phi^{\xi_i}$. Then $\Phi = (a_1, \dots, a_d): Z \rightarrow \mathbb{R}^d$.

Let us also write $\theta = (\theta_1, \dots, \theta_d)$ for the associated coordinates on \mathfrak{t} and $e(\theta, \dots, \theta_d) = (e^{2\pi i \theta_1}, \dots, e^{2\pi i \theta_d}) \in T^d$.

We will also make the assumptions in [13]:

- (a) 0 is not in the image of $\Phi: Z \rightarrow \mathfrak{t}^*$.
- (b) There is an open dense subset Z_0 of Z on which the action is free.

Under these conditions, the total action can be quantized; i.e., there exists a representation $\rho: T^d \rightarrow UF(M)$ by unitary FIOs (Fourier Integral Operators) so that the canonical transformation underlying $\rho(e^{i\theta})$ is the graph of $\gamma(e^{i\theta}, \cdot)$ (see [13, 7]). In more concrete terms, there are d commuting ψDO 's of order 1, $\{A_1, \dots, A_d\}$ so that $\sigma_{A_i} = a_j$ and so that $\rho(e^{i\theta}) = e^{2\pi i(\theta_1 A_1 + \dots + \theta_d A_d)}$.

DEFINITION 3.3. Let G' be a geodesic flow on Z . Then G' is *collectively completely integrable* if there exists a homogeneous hamiltonian torus action which commutes with G' , and indeed such that the riemannian norm function $|z|$ equals $|\Phi|$ ($|\cdot|$ being a Euclidean norm on \mathfrak{t}^*).

From now on we assume that M has a collectively completely integrable geodesic flow.

Let $L^2(M) = \bigoplus_{\alpha} \mathcal{H}_{\alpha}$ be the decomposition of M into isotropic subspaces for the associated ρ , with α an integral form in \mathfrak{t}^* and $\mathcal{H}_{\alpha} = \{f \in L^2: \rho(e^{i\theta}) = e^{2\pi i \langle \alpha, \theta \rangle} f\}$. Except for finitely many α , $\dim \mathcal{H}_{\alpha} = 1$ [13, 1]. Let Φ_{α} be a unit vector in \mathcal{H}_{α} .

Following [7] one calls a subspace of the form $\mathcal{H}_{L\alpha} = \bigoplus_{k=1}^{\infty} \mathcal{H}_{k\alpha}$ a "ladder." \mathcal{H}_{α} corresponds to a conic co-isotropic submanifold W_{α} of Z : namely, $\Phi^{-1}\{r\alpha: r \in \mathbb{R}^+\}$. W_{α} is foliated by leaves on which the symplectic form of Z vanishes ("null-foliation") and it fibers over the leaf space $X_{\alpha}^{\#}$, a symplectic manifold. Thus, there is a fibration $\pi: W_{\alpha} \rightarrow X_{\alpha}^{\#}$ so that $\pi^{-1}(x)$ is the null leaf through any of its points. Then $\mathcal{H}_{L\alpha}$ corresponds to W_{α} in the precise sense that the orthoprojection $\Pi_{\alpha}: L^2(M) \rightarrow \mathcal{H}_{L\alpha}$ is an FIO whose underlying canonical relation is the fiber product $\Gamma_{\alpha} = W_{\alpha} \times_{\pi} W_{\alpha} = \{(w_1, w_2): \pi(w_1) = \pi(w_2)\}$.

As observed by Uribe (and independently by the author in special cases), the family $\mathcal{F}_{\alpha,0} = \{d\Phi_{k\alpha,k\alpha}: k \in \mathbb{Z}^+\}$ is coherent and its classical limit is the invariant δ -measure dv_{α} along W_{α}/\mathbb{R}^+ (a Lagrangian torus). The family $\mathcal{F}_{\alpha,0}$ is one of many similar families.

DEFINITION 3.4. For integral forms $\alpha, \beta \in t^*$, let $\mathcal{F}_{\alpha, \beta} = \{d\Phi_{k\alpha, k\alpha + \beta} : k \in \mathbb{Z}^+\}$.

We now must point out one arbitrary feature in the definition of $\mathcal{F}_{\alpha, \beta}$ which plays no role for diagonal families, nor in the ergodic case (because “fat” classical limits were 0). Namely, the ϕ_α are only determined up to scalars $e^{i\theta_\alpha}$. It follows that the $d\Phi_{k\alpha, k\alpha + \beta}$ are only determined up to the units $e^{i(\theta_{k\alpha} - \theta_{k\alpha + \beta})}$. Consequently, only the limits of $|\langle \sigma, d\Phi_{k\alpha, k\alpha + \beta} \rangle|$ are invariants of a ladder.

This ambiguity appears as well on the side of the classical limits. Indeed consider the measures $\mu_{z_0, \alpha}^\beta$ which give fourier coefficients along the tori $W_x^1 = \{w \in W_x : |w| = 1\}$, for $f \in C^\infty(S^*M)$,

$$\mu_{z_0, \alpha}^\beta(f) = \frac{1}{\text{vol}(T^d)} \int_{T^d} e^{-2\pi i \langle \beta, \theta \rangle} f(\gamma(e^{i\theta}, z_0)) d\theta. \quad (3.1)$$

It is clear that $\mu_{z_0, \alpha}^\beta$ depends on z_0 , but that $|\mu_{z_0, \alpha}^\beta|$ does not.

One has two ways of dealing with this arbitrariness. The first is to look only at limits of $|(A\Phi_i, \Phi_j)|$, and this leads to Theorem D. The second is to choose the vectors $\Phi_{k\alpha}$ and $\Phi_{k\alpha + \beta}$ in some coherent way for each fixed ladder $H_{L\alpha}$. This can be done by using a local normal form and leads to coherent families and classical limits which are uniquely determined up to a fixed scalar $e^{i\theta_0}$.

THEOREM D. With the above notation and assumptions, for any 0th order $\psi D0 A$, $(1/N) \sum_{k=1}^N |(A\Phi_{k\alpha}, \Phi_{k\alpha + \beta})|^2 \rightarrow |(1/\text{vol}(T^d)) \int_{T^d} \sigma_A(\gamma(e^{i\theta}, z_0)) e^{-i \langle \beta, \theta \rangle} d\theta|^2$ for any $z_0 \in W_\alpha^1$.

Proof. Consider the operators

$$A_{\lambda, \alpha, f} = \pi_\lambda A_f \Pi_\alpha \pi_\lambda, \quad (3.2)$$

where $A_f = (1/\text{vol}(T^d)) \int_{T^d} f(e^{i\theta}) \rho(\theta)^* A \rho(\theta) d\theta$. Then

$$\text{Tr } A_{\lambda, \alpha, f}^* A_{\lambda, \alpha, f} = \sum_{\substack{k \mid \alpha| \leq \lambda \\ |\delta| \leq \lambda}} |(A\Phi_{k\alpha}, \Phi_\delta)|^2 |\hat{f}(k\alpha - \delta)|^2. \quad (3.3)$$

It follows that if $f = \overline{\chi_\beta}$ and if $A_{\lambda, \alpha, \beta}$ is the corresponding operator, then

$$\text{Tr } A_{\lambda, \alpha, \beta}^* A_{\lambda, \alpha, \beta} = \sum_{k \mid \alpha| \leq \lambda} |(A\Phi_{k\alpha}, \Phi_{k\alpha + \beta})|^2. \quad (3.4)$$

Thus the left side of our purported limit formula is $\lim_{\lambda \rightarrow \infty} (1/\text{rank}(\pi_\lambda \Pi_\alpha)) \text{tr } A_{\lambda, \alpha, \beta}^* A_{\lambda, \alpha, \beta}$. We can compute this symbolically by a slight modification of Widom's argument in (1.9)–(1.10). Indeed we claim that

$$\text{tr } A_{\lambda, \alpha, \beta}^* A_{\lambda, \alpha, \beta} = \text{tr } \pi_\lambda \Pi_\alpha A_f^* A_f \Pi_\alpha \pi_\lambda + o(\text{rank } \pi_\lambda \Pi_\alpha). \quad (3.5)$$

The proof is essentially identical to the standard one, and is left to the reader. We then note that $\Pi_\alpha A_f^* A_f \Pi_\alpha$ is an FIO associated to the same canonical relation $\Gamma_\alpha = W_\alpha \times_\pi W_\alpha$ as Π_α . The principal symbol of A_{χ_β} is (for $z_0 \in Z$)

$$\sigma_{A_{\chi_\beta}}(z_0) = \frac{1}{\text{vol}(T^d)} \int_{T^d} \sigma_A(\gamma(e^{i\theta}, z_0)) e^{-2\pi i \langle \beta, \theta \rangle} d\theta, \quad (3.6)$$

hence that of $A_{\chi_\beta}^* A_{\chi_\beta}$ ($|\sigma_{A_{\chi_\beta}}(z_0)|^2$) is actually constant on every orbit of the torus action. To describe the principal symbol of $\Pi_\alpha A_f^* A_f \Pi_\alpha$, we will need to recall the theory of such symbols in [7, 8]. Thus, let Σ be any homogeneous, co-isotropic submanifold of T^*M which is fibering over the leaf space S of its null-foliation (S is a symplectic manifold). Letting $\pi: \Sigma \rightarrow S$ be this fibration, set $\Gamma = \Sigma \times_\pi \Sigma$. (The example of interest to us is $\Sigma = W_\alpha$, $S = X_\alpha^*$, $\Gamma = \Gamma_\alpha$.) Let R_Σ denote the ($*$ algebra of) FIOs associated to Γ . Guillemin and Sternberg interpret symbols of $A \in R_\Sigma$ in terms of a certain sheaf $\mathcal{S}\mathcal{C}$ of $*$ -algebras over S . Namely, $\mathcal{S}\mathcal{C}_s$ is the $*$ -algebra of smoothing operators $\kappa: C^\infty(|A|^{1/2} F_s) \rightarrow C^\infty(|A|^{1/2} F_s)$, where F_s is the fibre over s . A section of the sheaf is given by a smooth kernel $\kappa(s; x, y)$ which for fixed s is a smooth $1/2$ density on $F_s \times F_s$.

Now the principal symbol of $\Pi_\alpha A_f^* A_f \Pi_\alpha$ is just $\sigma_{\Pi_\alpha} \cdot |\sigma_{A_f}|^2 \cdot \sigma_{\Pi_\alpha}$. If $f = \chi_\beta$ this is $|\sigma_{A_{\chi_\beta}}|^2 \sigma_{\Pi_\alpha}$ since $|\sigma_{A_{\chi_\beta}}|^2$ is multiplication by a constant. σ_{Π_α} is computed in [7]: $\sigma_{\Pi_\alpha}(s)$ is projection onto the invariant $1/2$ density $|dv_s|^{1/2}$ in $C^\infty(|A|^{1/2} F_s)$. In sum, $\sigma_{\Pi_\alpha A_{\chi_\beta}^* A_{\chi_\beta} \Pi_\alpha}(s) = |\sigma_{A_{\chi_\beta}}|^2 |dv_s|^{1/2} \otimes |dv_s|^{1/2}$.

It follows that

$$\frac{1}{\text{tr } \pi_\lambda \Pi_\alpha} \text{tr } \pi_\lambda \Pi_\alpha A_{\chi_\beta}^* A_{\chi_\beta} \Pi_\alpha \pi_\lambda \rightarrow |\sigma_{A_{\chi_\beta}}|^2 (W_\alpha^1), \quad (3.7)$$

where the constant at the right means the value $|\sigma_{A_{\chi_\beta}}(z_0)|^2$ at one (hence all) $z_0 \in W_\alpha^1$. Indeed, the (standard) argument is to note that the limit of the left side of (3.7) equals the coefficient of the singularity at $t=0$ of

$$S_{\alpha\beta}(t) = \text{Tr } e^{itD} \Pi_\alpha A_{\chi_\beta}^* A_{\chi_\beta} \Pi_\alpha, \quad (3.8)$$

where D is a first order self-adjoint $\psi D 0$ equal to k on $\mathcal{H}_{k\alpha}$. In turn this coefficient is the principal symbol of the Fourier Integral distribution $S_{\alpha\beta}(t)$, evaluated at $t=0$ and $\tau=1$ ($T^*\mathbb{R} = \{(1, \tau)\}$). Writing $S_{\alpha\beta}(t) = \pi_0^* A^* e^{itD} \Pi_\alpha A_{\chi_\beta}^* A_{\chi_\beta} \Pi_\alpha$ ($\pi_0: M \times \mathbb{R} \rightarrow \mathbb{R}$, $A: M \times \mathbb{R} \rightarrow M \times M \times \mathbb{R}$ the usual maps [7, 13]), one sees that the principal symbol of $S_{\alpha\beta}(t)$ at $t=0$, $\pi=1$

is given by integrating $\text{tr } \sigma_{\Pi_\alpha} \cdot \sigma_{|A_{\alpha\beta}|^2}(s)$ over S/\mathbb{R} relative to the invariant density on S/\mathbb{R} (insert the cone direction vector into the symplectic volume form). By our computation above this is just $|\sigma_{A_{\alpha\beta}}|^2$. This proves (3.7), hence the theorem. ■

Theorem D has one unsatisfactory aspect: namely, it doesn't prove that the terms individually tend to the average value. Yet this is of course what is meant by a classical limit of a coherent family. This defect can be remedied:

THEOREM E. *For a given α , unit vectors $\Phi_{k\alpha+\beta}$ in $\mathcal{H}_{k\alpha+\beta}$ can be chosen in such a way that $\mathcal{F}_{\alpha,\beta} = \{d\Phi_{k\alpha,\alpha+\beta}\}$ is a coherent family. Moreover, its classical limit is given by an orbital fourier coefficient $\mu_{z_0,\alpha}^\beta$ for some $z_0 \in W_\alpha^1$ (depending on the choice of $\{\Phi_{k\alpha+\beta}\}$).*

Proof. The existence of such a coherent basis and the computation of the classical limit follows from Colin de Verdiere's microlocal norm theorem for quantizations of symplectic toral actions. We begin with a resume of Colin de Verdiere's work.

Let (Z, ω) be a conic symplectic manifold of dimension $2d$, $\gamma: T^d \times Z \rightarrow Z$ be a conic symplectic toral action, $z_0 \in Z$, and $Y = T^d \cdot z_0$ be the orbit through z_0 . Y is diffeomorphic to the torus $T^d/T_{z_0}^d$ where $T_{z_0}^d$ is the stabilizer of z_0 . Suppose $\dim(T_{z_0}^d) = d-l$ and consider

$$\begin{aligned} T^*(T^l \times \mathbb{R}^{d-l}) \\ = \{(x_1, \dots, x_l; \xi_1, \dots, \xi_l; y_{l+1}, \dots, y_d; \eta_{l+1}, \dots, \eta_d) \\ \in T^l \times \mathbb{R}^l \times \mathbb{R}^{d-l} \times \mathbb{R}^{d-l}\}. \end{aligned} \quad (3.9)$$

Let Z_0 be the conic open subset

$$Z_0 = \{(x, \xi, y, \eta) : \xi \neq 0\}. \quad (3.10)$$

$T^*(T^l \times \mathbb{R}^{d-l})$ carries a linear symplectic toral action which is a model for symplectic toral actions near an orbit whose isotropy group equals T^{d-l} . Namely, write $z_j = y_j + i\eta_j$ and let $\beta_0(e^{i(\theta', \theta'')}) \cdot (x, \xi, y + i\eta) = (x + \theta', \xi, e^{i\theta''}(y + i\eta))$, where $\theta' = (\theta_1, \dots, \theta_l)$, $\theta'' = (\theta_{l+1}, \dots, \theta_d)$ and where an obvious multi-index notation is assumed. β_0 can be modified to provide a local model for homogeneous total actions in a conic neighborhood of an orbit. Indeed, this is done by homogenizing the Hamiltonians $\tilde{q}_j^0 = \xi_j$ ($j = 1, \dots, l$), $\tilde{q}_j^0 = (1/2)(y_j^2 + \eta_j^2)$ ($j = l+1, \dots, d$) for the linear action. One sets, for $(x, \xi, y, \eta) \in Z_0$,

$$\begin{aligned} q_j^0(x, \xi, \eta) &= \xi_j & (j = 1, \dots, l) \\ &= \frac{1}{2}(|\xi| y_j^2 + |\xi|^{-1} \eta_j^2) & (j = l+1, \dots, d). \end{aligned} \quad (3.11)$$

Each q_j^0 generates a Hamiltonian flow of period 2π ; since $\{q_j^0, q_k^0\} = 0$, they collectively generate a T^d action γ_0 . Let $\Phi^0 = (q^0, \dots, q_d^0)$; Φ^0 is the moment map for the toral action. It is clear that orbits are level sets of Φ^0 , and that on $Z_0 \cap \{(y, \eta) \neq 0\}$ the action is free.

One has the following theorem due to [1]:

THEOREM 3.1. *With the notations above, and given an orbit $Y = T^d z_0$, there is a conic, invariant open subset U of Z and a conic canonical diffeomorphism $\chi: U \rightarrow U_0$, U_0 an invariant conic neighborhood of $(\mathbb{R}/2\pi\mathbb{Z})^l \times \{0\}$ in Z_0 , so that*

$$\begin{array}{ccc} T^d \times U & \xrightarrow{\gamma} & U \\ \rho_0 \times \chi \downarrow & & \downarrow \chi \\ T^d \times U_0 & \xrightarrow{\gamma_0} & U_0 \end{array}$$

commutes; here ρ_0 is an isomorphism: $T^d \rightarrow T^d/T_{z_0}^d \times T_{z_0}^d$.

Furthermore, Colin de Verdiere has proved a quantized version of this [1, Sect. 5]. To state it, we need some more notation, and background:

- (i) $C_\alpha = a$ a conic neighborhood of $\{r\alpha : r > 0\}$ so that the normal form Theorem 3.1 is valid.
- (ii) ξ_1, \dots, ξ_d is a basis for t so that $\phi^{\xi_i} = \text{def } q_i$ have periodic Hamiltonian flows of period 2π (they exist). Let Q_1, \dots, Q_d be first order $\psi D0$'s, commuting $\psi D0$'s with principal symbols $\sigma_{Q_i} = q_i$ (they exist).
- (iii) $E_\alpha = \{\Sigma a_\lambda \Phi_\lambda \in L^2(X) : \lambda \in C_\alpha\}$.
- (iv) $X_0 = (\mathbb{R}/2\pi\mathbb{Z})^l \times \mathbb{R}^{d-l}$.
- (v) $\tilde{Q}_j^0 = (1/i)(\partial/\partial x_j)$ ($1 \leq j \leq l$); $\tilde{Q}_j^0 = (1/2)(-\partial^2/\partial y_j^2 + y_j \Delta_x)$ ($\Delta_x^{-1/2}$); $\Delta_x = -\Sigma_{j=1}^l (\partial^2/\partial x_j^2)$ ($l+1 \leq j \leq d$).
- (iv) $\tilde{\Phi}_\lambda^0$ are the following joint eigenfunctions for $\{\tilde{Q}_j^0\}$: For $\lambda = (n, n') \in \mathbb{Z}^l \times \mathbb{N}^{d-l}$, let $\tilde{\Phi}_\lambda^0(x_1, \dots, y_d) = \exp(n_1 x_1 + \dots + n_l x_l), h_{n_{l+1}}(\sqrt{\|n\|} y_{l+1}) \dots h_{n_d}(\sqrt{\|n\|} y_d)$. Here $h_n(t) = H_n(t) e^{t^2/2}$, H_n being the n th Hermite polynomial.
- (vii) Let $E_\alpha^0 = \{\Sigma a_\lambda \Phi_\lambda^0 \in L^2(X_0) : \lambda \in C_\alpha\}$. Let $\mathcal{H}_{L_\alpha}^0$ be the ladder through ϕ_α^0 .

(3.12)

Then Colin de Verdiere's quantized normal form theorem states:

THEOREM 3.2. *With the above notation and assumptions, there exists a 0th order FIO*

$$F_x: L^2(X) \rightarrow L^2(X_0),$$

F_x elliptic on $\Phi^{-1}(C_x)$, with underlying canonical transformation χ , and there exists a vector $\mu' = (\mu'_1, \dots, \mu'_l, 0, \dots, 0) \in (1/4)\mathbb{Z}^d$ satisfying:

- (a) $F_x^* F_x - \text{Id}$ (resp. $F_x F_x^* - \text{Id}$) is of order -1 on E_x (resp. E_x^0)
- (b) $(\tilde{Q}_j^0 + \mu'_j) F_x - F_x \tilde{Q}_j$ (resp., $F_x^*(\tilde{Q}_j^0 + \mu'_j) - Q_j F_x^*$) is of order -1 on E_x (resp. E_x^0) for $j = 1, \dots, d$
- (c) if B is a (-1st) order FIO on E_x , then $\|B\Phi_\lambda\|_{L^2(X)} = O(\|\lambda\|^{-1}) \|\Phi_\lambda\|_{L^2(X)}$, $\lambda \in C_x$ (similarly for E_x^0).

Actually, it will be convenient for us to modify this theorem on normal forms for homogeneous toral actions and their quantizations. Namely, we observe that the original linear action β_0 on $T^*(T^l \times \mathbb{R}^{d-l})$ is itself homogeneous when $T^*(T^l \times \mathbb{R}^{d-l})$ is given the conic structure

$$\tau \cdot (x, \xi, y, \eta) = (x, t\xi, \tau^{1/2}y, \tau^{1/2}\eta), \quad \tau \in \mathbb{R}^+. \quad (3.13)$$

Moreover, we observe that β_0 is conically canonically conjugate to the homogeneous toral defined in (3.11). Indeed, a simple computation shows that the map

$$\psi: (x, \xi, y, \eta) \rightarrow (x - (y\eta) |\xi|^{-1}, \xi, y |\xi|^{-1/2}, \eta |\xi|^{-1/2}) \quad (3.14)$$

is symplectic; and it clearly intertwines the conic structures, and the Hamiltonians $\{\tilde{q}_j^0\}$ and $\{q_j^0\}$. So we may use β_0 as our homogeneous model.

The quantized version of this raises a problem, presumably explaining why Colin de Verdiere chose to use the normal form in (3.11). Namely, the operators

$$\begin{aligned} Q_j^0 &= \frac{1}{i} \frac{\partial}{\partial x_j} \quad j = 1, \dots, l \\ Q_j^0 &= \frac{1}{2} \left(-\frac{\partial^2}{\partial y_j^2} + y_j^2 \right) \end{aligned} \quad (3.15)$$

are no longer standard $\psi D0$'s; similarly, the conjugating FIOs F_x (etc.) are no longer standard. Rather they are $\psi D0$'s whose symbols satisfy differentiation conditions relative to the conic structure (3.13). However, the theory of these $\psi D0$'s is now very well known [9, 8], and exactly parallels the standard $\psi D0$ theory, at least insofar as we need it. We can therefore

use (3.15) as our model example, and we now turn to the proof of Theorem E proper.

(I) We start by proving the theorem for the model cases. Thus, consider the limits of $\langle A\Phi_{\lambda_r}^0, \Phi_{\lambda'_r}^0 \rangle$ where (as in (3.12(vi)))

$$\begin{aligned}\lambda &= (n, n'), \Phi_{\lambda}^0(x_1, \dots, y_d) \\ &= e^{i(n_1 x_1 + \dots + n_l x_l)} h_{n'_{l+1}}(y_{l+1}) \cdots h_{n'_d}(y_d).\end{aligned}\quad (3.16)$$

We claim the corresponding family $\{d\Phi_{\lambda_r, \lambda'_r}^0\}$ is a coherent family iff $\lambda_r/|\lambda_r| \rightarrow \alpha$ and $\lambda_r - \lambda'_r = \beta$ for some $\beta \in \mathbb{Z}^d$ and for all but finitely many r (these are of course necessary conditions).

Indeed, first observe that if $\lambda - \lambda' = \beta$, then A may be replaced by $A_\beta = \int_{T^d} \chi_\beta(\theta) \rho(\theta)^* A \rho(\theta) d\theta$ without changing $\langle A\Phi_{\lambda}^0, \Phi_{\lambda'}^0 \rangle$. But

$$\sigma_{A_\beta}(x, \xi, y, \eta) = \int_{T^d} \overline{\chi_\beta(\theta)} \sigma_A(\gamma(e^{i\theta}, (x, \xi, y, \eta))) d\theta$$

is homogeneous of order 0 and transforms by the character χ_β under the torus action. Let \mathcal{A}_β denote the space of non-negatively homogeneous functions transforming this way. \mathcal{A}_0 in particular is the algebra of homogeneous, invariant functions. Colin de Verdiere has characterized \mathcal{A}_0 in [1] as

$$\mathcal{A}_0 = \{f(q_1^0, \dots, q_d^0) : f \in C^\infty(\mathbb{R}|0), f \text{ non-negatively homogeneous}\}. \quad (3.17)$$

It follows that if $\beta = 0$, $\sigma_{A_\beta} = f(q_1^0, \dots, q_d^0)$ for some function f , homogeneous of degree 0. By a well-known argument of Strichartz (cf. [1]), $f(Q_1^0, \dots, Q_d^0)$ as defined by the spectral theorem is a 0th order ψDO (in the relevant calculus) with principal symbol $f(q_1^0, \dots, q_d^0)$. Hence, $\langle A\Phi_{\lambda}^0, \Phi_{\lambda'}^0 \rangle$ has the same limits as $\langle f(Q_1^0, \dots, Q_d^0) \phi_{\lambda}^0, \phi_{\lambda'}^0 \rangle = f(\lambda)$. A necessary and sufficient condition that $\lim_{\lambda \rightarrow \infty} f(\lambda)$ exists for homogeneous f of order 0 is that $\lambda/|\lambda| \rightarrow \alpha$ for some α , and the limit is then $f(\alpha)$. $f(\alpha)$ also equals the value of f on W_α^1 , hence the average of σ_A on W_α^1/\mathbb{R}^+ . We conclude that $\langle A\Phi_{\lambda}^0, \Phi_{\lambda}^0 \rangle$ has a limit iff $\lambda/|\lambda| \rightarrow \alpha$, and the limit is then $\sigma_{A_0}(W_\alpha)$ (i.e., the value of σ_{A_0} at once, hence any, point of W_α). This confirms the theorem for model diagonal sequences.

For an off-diagonal sequence the averaged symbols belong to \mathcal{A}_β , $\beta \neq 0$. Our first step is to extend Colin de Verdiere's characterization of \mathcal{A}_0 , to $\mathcal{A}_\beta \neq 0$. To this end we need to construct special elements $e_\beta^0 \in \mathcal{A}_\beta$,

$$e_\beta^0(x, \xi, y, \eta) = e^{2\pi i \langle x, \beta' \rangle} \prod_{j=l+1}^d \left(\frac{y_j + i\eta_j}{|y_j + i\eta_j|} \right)^{\beta_j''}, \quad (3.18)$$

where $\beta = (\beta', \beta'') \in \mathbb{R}^l \times \mathbb{R}^{d-l}$. If $\beta_j'' = 0$, one omits the corresponding factor (i.e., it equals 1 by definition).

e_β^0 is well defined away from set $\Sigma = U_j \Sigma_j$, $\Sigma_j = \{(x, \xi, y, \eta) : y_j + i\eta_j = 0\}$, and is clearly homogeneous of order 0 and nowhere vanishing on $Z \setminus \Sigma$. Note that any $a_\beta \in \mathcal{A}_\beta$ must vanish on Σ_j if $\beta_j'' \neq 0$; so e_β is in a sense optimally defined.

Now let $\mathcal{A}_0^{\beta'} = \{a \in \mathcal{A}_0 : a \text{ vanishes to order 1 on } \Sigma_\beta = U_{j: \beta_j'' \neq 0} \Sigma_j\}$. Then

$$\mathcal{A}_\beta = \mathcal{A}_0^{\beta'} e_\beta^0. \quad (3.19)$$

Indeed if $a_\beta \in \mathcal{A}_\beta$, then a_β vanishes to order 1 on Σ_β , so that $a_\beta (e_\beta^0)^{-1}$ is a well-defined element of $\mathcal{A}_0^{\beta'}$. Conversely it is clear that $\mathcal{A}_0^{\beta'} e_\beta^0 \subseteq \mathcal{A}_\beta$.

It therefore follows from Colin de Verdière's description of \mathcal{A}_0 that

$$a_\beta \in \mathcal{A}_\beta \Rightarrow a_\beta = f(q_1^0, \dots, q_d^0) e_\beta, \quad (3.20)$$

where f vanishes to order 1 on $\{q_j^0 = 0 : \beta_j'' \neq 0\}$.

Our second step is to quantize this. But it is clear that

$$\text{Op}(e_\beta^0) = e^{2\pi i \langle x, \beta' \rangle} \prod_{j=1}^d \left| -\frac{\partial^2}{\partial y_j^2} + y_j^2 \right|^{-\beta_j''} \left(\frac{1}{i} \frac{\partial}{\partial y_j} + y_j \right)^{\beta_j'} \quad (3.21)$$

$\text{Op}(e_\beta)$ is thus a normalized raising operator by β units, familiar from Harmonic oscillator theory.

It follows easily that

$$(A_\beta \Phi_\lambda^0, \Phi_{\lambda'}^0) \sim (f(Q_1^0, \dots, Q_d^0) \Phi_\lambda^0, \Phi_{\lambda'}^0), \quad (3.22)$$

where $\sigma_{A_\beta} = f e_\beta^0$.

Summing up:

$$\text{If } \lambda' = \lambda + \beta, \frac{\lambda}{|\lambda|} \rightarrow \alpha, \text{ then } (A \Phi_\lambda^0, \Phi_{\lambda'}^0) \rightarrow f(\alpha), \text{ where } \sigma_{A_\beta} = f e_\beta^0. \quad (3.23)$$

More invariantly put, $e_\beta^0 \equiv 1$ on the set $N = \{(0, \xi, y, 0) : y_j \geq 0\}$. N is a slice of the torus action, so e_β^0 could have been defined as the extension of 1 to $Z \setminus \Sigma$ as a function transforming by χ_β . In any event, $f(\alpha)$ equals σ_{A_β} on $W_\alpha \cap N$. We thus have

$$(A \Phi_\lambda^0, \Phi_{\lambda'}^0) \rightarrow \sigma_{A_\beta}(W_\alpha \cap N), \quad (3.24)$$

where the evaluation at right means again at one, hence any point in $W_\alpha \cap N$. This confirms the theorem for the model cases.

Remark. The coherency of the basis $\{\Phi_{\lambda_j}^0\}$ of (3.16) is now seen to reside in the fact that higher Φ_λ^0 come from lower by applying the normalized raising-lowering operators (3.21). This clearly fixes the family up to

a fixed unit scalar, and the limits will be independent of that scalar. Note also that the reason why the limit in (3.24) involves a special slice N of the action is that we have selected special raising-lowering operators to define the basis (3.16).

(II) We next extend the theorem from the model cases to all cases.

Thus let α be a weight of the quantized toral action, and let C_α be the conic neighborhood of $\{r\alpha: r > 0\}$ where the normal form theorems are valid (3.12(i)). The subspace E_α is then well defined ((3.12(ii))), and one has an FIO F_α conjugating the given Q_j 's on X to the Q_j^0 's on X_0 , modulo operators of lower order (by means of the modified Theorem 3.2).

By means of F_α one can construct coherent orthonormal bases $\{\Phi_\lambda\}$ for E_α .

Indeed, let \mathcal{H}_λ^0 and \mathcal{H}_λ be the 1-dimensional joint eigenspaces defined previously. Let Φ_λ^0 be the unit vectors in H_λ^0 from (3.16), and let ψ_λ be any unit vector in \mathcal{H}_λ . We set:

DEFINITION 3.5. Let $\{\Phi_\lambda, \lambda \in C_\alpha\}$ be the orthonormal basis for E_α given by $\Phi_\lambda = \langle F_\alpha^* \Phi_\lambda^0, \psi_\lambda \rangle \psi_\lambda / |\langle F_\alpha^* \Phi_\lambda^0, \psi_\lambda \rangle|$.

We will see below that $|\langle F_\alpha^* \Phi_\lambda^0, \psi_\lambda \rangle| = 1 + O(|\lambda|^{-1})$, so that the definition makes sense.

A simple consequence of the spectral theorem is:

PROPOSITION 3.3. $\|F_\alpha^* \Phi_\lambda^0 - \langle F_\alpha^* \Phi_\lambda^0, \psi_\lambda \rangle \psi_\lambda\| = O(\|\lambda\|^{-1})$.

Proof. From Theorem 3.2(b), we see that

$$QF_\alpha^* \Phi_\lambda^0 = (\lambda + \mu) F_\alpha^* \Phi_\lambda^0 + O(|\lambda|^{-1}).$$

However, $Q\psi_\lambda = (\lambda + \mu) \psi_\lambda$, $\lambda \in C_\alpha$ [1, Theorem 3.2].

Hence $f_\lambda = F_\alpha^* \Phi_\lambda^0 - \langle F_\alpha^* \Phi_\lambda^0, \psi_\lambda \rangle \psi_\lambda \in E_\alpha \ominus \mathcal{H}_\lambda$ and $[Q - (\lambda + \mu)]f_\lambda = O(|\lambda|^{-1})$. But the resolvent $R_{\lambda+\mu}(Q) = (Q - (\lambda + \mu))^{-1}$ is bounded on $E_\alpha \ominus \mathcal{H}_\lambda$ by the distance from $\lambda + \mu$ to the nearest spectral point, hence by a constant independent of λ . It follows that $\|f_\lambda\| = O(|\lambda|^{-1})$. ■

COROLLARY 3.4. $|\langle F_\alpha^* \Phi_\lambda^0, \psi_\lambda \rangle| = 1 + O(|\lambda|^{-1})$.

Proof. $\|F_\alpha^* \Phi_\lambda^0\| = 1 + O(|\lambda|^{-1})$ by Theorem 3.2. ■

We now complete the proof of Theorem E. For a given α , we have just defined an orthonormal basis of E_α . Clearly all vectors of the form $\Phi_{k\alpha + \beta}$,

$k = 1, 2, \dots$, live in E_x , so the families $\mathcal{F}_{x,\beta}$ are well defined. The classical limit is by the propositions above exactly that of

$$\begin{aligned}
 & (AF_x^* \Phi_{kx}^0, F_x^* \Phi_{kx+\beta}^0) \\
 &= (F_x AF_x^* \Phi_{kx}^0, \Phi_{kx+\beta}^0) \\
 &= (\text{Op}(\sigma_A \cdot \chi_x^{-1}) \Phi_{kx}^0, \Phi_{kx+\beta}^0) + O(k^{-1}) \\
 &\rightarrow (\sigma_A \cdot \chi_x^{-1})(W_x^0 \cap N) \quad \text{by (3.24)} \\
 &= \sigma_{A\beta}(W_x \cap \chi_x^{-1}N).
 \end{aligned}$$

This is the orbital fourier coefficient claimed in the theorem.

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